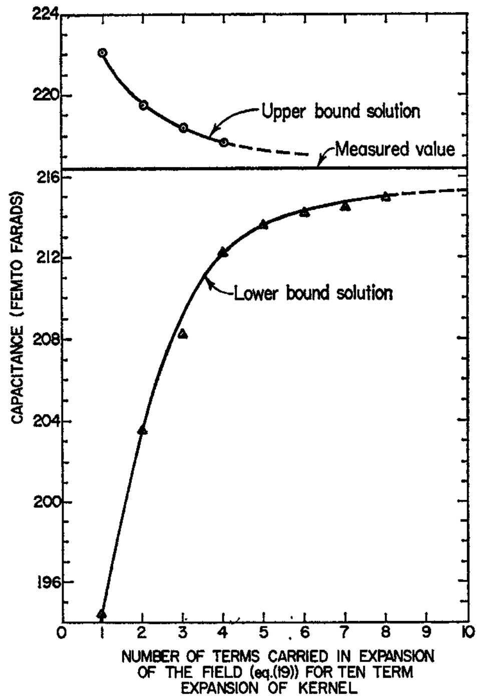

Fig. 2. Convergence behavior of upper and lower bound solutions.

where

$$\delta_{inj} = \begin{cases} 1, & i = n = j \\ 0, & \text{otherwise.} \end{cases}$$

IV. Results

Equation (21) was evaluated using an electronic computer. A 50-Ω 0.9525-cm (3/4-in) open-circuited coaxial termination with a solid center conductor was fabricated with center- and outer-conductor diameters of 0.82723 ± 0.00005 and 1.90487 ± 0.00005 cm (1 cm = 0.393701 in), respectively. The measured value of capacitance of this termination at 1000 Hz was 216.4 ± 1.0 fF, as compared with the calculated lower bound of 215.0 fF. [The upper bound for this case was 217.7 fF (see 2).] The number of terms carried in the expansions for H and $k(\rho, \rho')$ were eight and ten, respectively.

Fig. 2 is a plot of the calculated value of capacitance as a function of the number of terms carried in the expansion of the field [see (19)] for a ten-term expansion of the kernel [see (17)]. Also displayed is the convergence behavior of the upper bound solution.

The error bounds provided by this method make it useable for standards work. In other methods, error bounds must be inferred from the convergence behavior of the solution. The minimum error bounds determinable by this method must wait until funds become available. In theory, of course, this limit could be reduced to zero.

Somlo [3] obtained a value of 216.8 fF using a 40-term expansion of the series derived by Whinnery *et al.* [4]. This value lies between the upper and lower bounds obtained here.

V. Nomenclature

A_1, A_2, A_3	See Fig. 1.
E	Radial component of transverse electric field.
H	Transverse component of magnetic field.
R	Reflection coefficient.
a_0	Amplitude of the incident wave in region I.
h_n	$=\sqrt{k^2-\gamma_n^2}=i\alpha_n$; $\alpha_n=\sqrt{\gamma_n^2-k^2}$.
h_n'	$=\sqrt{k^2-\gamma_n'^2}=i\alpha_n'$; $\alpha_n'=\sqrt{\gamma_n'^2-k^2}$.
k	$=\omega\sqrt{\mu\epsilon}$.
y_n	$=\omega\epsilon/h_n$—wave admittance corresponding to the nth mode in the region $Z<0$.
y_n'	$=\omega\epsilon/h_n'$—wave admittance corresponding to the nth mode in the region $Z>0$.
y_0	Characteristic admittance.
$\psi_n(\rho)$	Mode function of the nth mode in a circular waveguide.
γ_n	nth eigenvalue corresponding to the eigenfunction $\Phi_n(\rho)$ (see I).
γ_n'	nth eigenvalue corresponding to the eigenfunction Ψ_n (see I).
ϵ	Dielectric constant.
ρ	Polar coordinate.
μ	Permeability.
φ	Polar coordinate.
$\varphi_0(\rho)$	Mode function of the dominant mode in a coaxial line.
$\phi_n(\rho)$	Mode function of the nth mode in a coaxial line.
ω	$=2\pi f$; $f=$frequency.

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References

[1] D. M. Kerns, "Half-round inductive obstacles in rectangular waveguide," *J. Res. Nat. Bur. Stand.*, vol. 64B, pp. 113–130, Jan. 1960.
[2] E. W. Risley, Jr., "Discontinuity capacitance of a coaxial line terminated in a circular waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 86–92, Feb. 1969.
[3] P. I. Somlo, "The discontinuity capacitance and the effective position of a shielded open circuit in a coaxial line," *Proc. IREE (Australia)*, vol. 28, no. 1, pp. 7–9, Jan. 1967.
[4] J. R. Whinnery, H. W. Jamieson, and T. E. Robbins, "Coaxial-line discontinuities," *Proc. IRE*, vol. 32, pp. 695–709, Nov. 1944.

On Inhomogeneously Filled Rectangular Waveguides

K. F. CASEY

***Abstract*—A method is given for determining the characteristic equations and field components of the LSE and LSM modes in rectangular waveguides filled with a dielectric which is inhomogeneous in one transverse dimension. The method is exact and yields solutions for a nearly arbitrary variation in permittivity across the waveguide.**

Propagation in waveguides which are inhomogeneously filled in the transverse direction has been of interest for many years, because of applications to a variety of microwave components, including phase changers, matching transformers, and quarter-wave plates [1]. In these applications, the inhomogeneous loading is generally accomplished by partially filling the guide cross section with a dielectric slab. There has also been some attention given to the more general problem in which the permittivity variation is continuous over one dimension of the guide cross section [2], [3]. In this short paper, we consider this more general situation and present a method by which the electromagnetic fields may be determined for a nearly arbitrary variation of permittivity across the waveguide.

Consider a rectangular waveguide formed by conducting surfaces at $x=0$ and $x=a$ and $y=0$ and $y=b$. The material filling the guide is an inhomogeneous dielectric of permittivity $\epsilon(x)$ and permeability μ_0. Assuming a time dependence exp $(j\omega t)$, the LSE modes are obtained from

$$\bar{E} = \nabla \times \Phi \bar{a}_x \tag{1a}$$

$$\bar{H} = \frac{1}{j\omega\mu_0} \nabla \times \nabla \times \Phi \bar{a}_x \tag{1b}$$

where

$$\nabla^2\Phi + k^2(x)\Phi = 0 \tag{2}$$

with $k^2(x) = \omega^2\mu_0\epsilon(x)$. The elementary product solutions of (2) are given by

$$\Phi(x, y, z) = f(x) \cos\frac{n\pi y}{b} e^{-j\beta z} \tag{3}$$

in which $n = 0, 1, 2, \cdots$, β is the propagation constant in the axial

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direction and $f(x)$ is a solution of

$$\frac{d^2f}{dx^2} + \left[k^2(x) - \left(\frac{n\pi}{b}\right)^2 - \beta^2\right]f = 0 \tag{4}$$

subject to the boundary conditions $f(0) = f(a) = 0$.

The LSM modes are obtained from

$$\bar{E} = -\frac{1}{j\omega\epsilon(x)}\nabla \times \nabla \times \Psi \bar{a}_x \tag{5a}$$

$$\bar{H} = \nabla \times \Psi \bar{a}_x \tag{5b}$$

where

$$\nabla^2\Psi - \frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{\partial \Psi}{\partial x} + k^2(x)\Psi = 0. \tag{6}$$

The elementary product solutions of (6) are given by

$$\Psi(x, y, z) = g(x)\sin\frac{n\pi y}{b}e^{-j\beta z} \tag{7}$$

in which $n = 1, 2, 3, \cdots$ and $g(x)$ satisfies

$$\frac{d^2g}{dx^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{dg}{dx} + \left[k^2(x) - \left(\frac{n\pi}{b}\right)^2 - \beta^2\right]g = 0 \tag{8}$$

subject to the boundary conditions $g'(0) = g'(a) = 0$.

Equations (4) and (8) may be solved via the substitutions

$$\xi = \frac{\pi x}{2a} \tag{9a}$$

$$u(\xi) = f(x) \tag{9b}$$

$$v(\xi) = \epsilon^{-1/2}(x)g(x) \tag{9c}$$

$$\lambda + 2\sum_{n=1}^{\infty} g_n \cos 2n\xi = \left(\frac{2a}{\pi}\right)^2\left[k^2(x) - \left(\frac{n\pi}{b}\right)^2 - \beta\right] \tag{9d}$$

$$\mu + 2\sum_{n=1}^{\infty} h_n \cos 2n\xi = \left(\frac{2a}{\pi}\right)^2 \epsilon^{1/2}(x)\frac{d^2}{dx^2}\left[\epsilon^{-1/2}(x)\right] \tag{9e}$$

yielding

$$\frac{d^2u}{d\xi^2} + \left(\lambda + 2\sum_{n=1}^{\infty} g_n \cos 2n\xi\right)u = 0 \tag{10a}$$

$$\frac{d^2v}{d\xi^2} + \left[\lambda - \mu + 2\sum_{n=1}^{\infty}(g_n - h_n)\cos 2n\xi\right]v = 0. \tag{10b}$$

Equation (10) is of the form of Hill's equation [4], [5], solutions of which may be readily obtained if the series of coefficients $\sum_{n=1}^{\infty} g_n$ (and $\sum_{n=1}^{\infty} h_n$ for the LSM fields) is absolutely convergent. The boundary conditions imposed on $u(\xi)$ and $v(\xi)$ are simply

$$u(0) = u(\pi/2) = 0 \tag{11a}$$

$$v'(0) = v'(\pi/2) = 0 \tag{11b}$$

when $\epsilon'(0) = \epsilon'(a) = 0$. If $\epsilon' \neq 0$ at either wall of the guide, condition (11b) will be modified somewhat.

The boundary condition equations (11) yield the characteristic equations for the propagation constants of the two mode types. One may readily show, using a theorem proved by Magnus [6] concerning the solutions to the Hill equation, that condition (11a) is equivalent to

$$\det\left\|\delta_{n,m} + \frac{g_{n-m} - g_{n+m}}{\lambda - 4n^2}\right\|_{n,m=1,2,3,\cdots} = 0. \tag{12a}$$

Condition (11b) is equivalent to

$$\det\left\|\delta_{n,m} + \frac{(g_{n-m} - h_{n-m} + g_{n+m} - h_{n+m})(1 + \operatorname{sgn} n \operatorname{sgn} m)}{\sqrt{\epsilon_m\epsilon_n}\,(\lambda - \mu - 4n^2)}\right\|_{n,m=0,1,2,\cdots} = 0. \tag{12b}$$

In (12), $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ if $n \neq m$; $\epsilon_n = 1$ if $n = 0$ and $\epsilon_n = 2$ if $n > 0$; and sgn $n = 1$ if $n > 0$ and sgn $0 = 0$. Equations (12a) and (12b) constitute the characteristic equations for the LSE and LSM modes, respectively; one will note that it has not been necessary to actually solve (4) or (8) to obtain them. Since these characteristic equations are expressed more or less directly in terms of the Fourier coefficients of the permittivity profile, the evaluation of the cutoff frequencies and propagation constants is numerically straightforward. The only condition imposed on the profile is that the series mentioned above must be absolutely convergent.

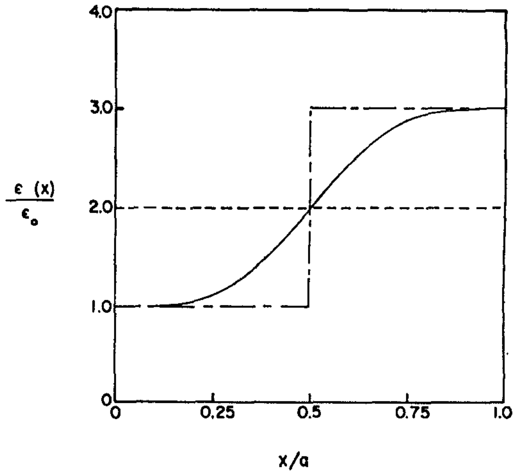

Fig. 1. Dielectric profiles. Solid line: $\epsilon(x)$ of (13); broken line: equivalent constant profile; dot-dashed line: step profile.

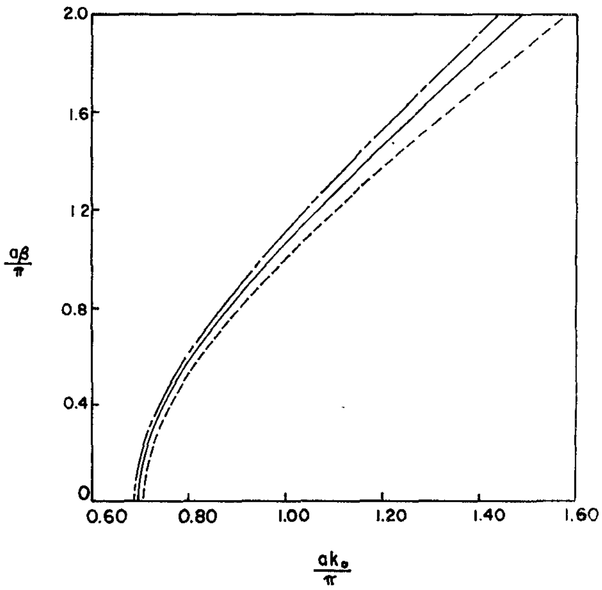

Fig. 2. Dominant-mode propagation constant $a\beta/\pi$ versus ak_0/π for the three dielectric profiles shown in Fig. 1.

As a simple example, consider the calculation of the dominant LSE-mode propagation constant in a rectangular guide filled with a medium of permittivity

$$\epsilon(x) = 2\epsilon_0 - \frac{9}{8}\epsilon_0\left(\cos\frac{\pi x}{a} - \frac{1}{9}\cos\frac{3\pi x}{a}\right). \tag{13}$$

$\epsilon(0) = \epsilon_0$ and $\epsilon(a) = 3\epsilon_0$; $\epsilon(x)/\epsilon_0$ is shown plotted versus x/a in Fig. 1. Also shown in Fig. 1 are the related constant and step profiles.

The normalized propagation constant of the dominant LSE mode, $a\beta/\pi$, is plotted as a function of normalized frequency ak_0/π ($k_0 = \omega\sqrt{\mu_0\epsilon_0}$) in Fig. 2. Also shown in Fig. 2 for comparison are curves of $a\beta/\pi$ versus ak_0/π for the homogeneously filled guide $\epsilon(x) = 2\epsilon_0$ and for the inhomogeneously filled guide $\epsilon(x) = \epsilon_0$ ($0 \leq x < a/2$) and $\epsilon(x) = 3\epsilon_0$ ($a/2 < x \leq a$).

References

[1] R. W. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960, ch. 6.

[2] D. A. Holmes, "Propagation in rectangular waveguide containing inhomogeneous, anisotropic dielectric," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-12, pp. 152–155, Mar. 1964.

[3] A. Wexler and D. A. Holmes, "Acceptable mode types for inhomogeneous media," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-13, pp. 875–878, Nov. 1965.

[4] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. London, England: Cambridge Univ. Press, 1963, ch. XIX.

[5] W. Magnus and S. Winkler, *Hill's Equation*. New York: Interscience, 1966.

[6] W. Magnus, "Infinite determinants associated with Hill's equation," *Pac. J. Math.*, vol. 5, suppl. 2, pp. 941–951, 1955.